

# Birkhoff's Theorem in Higher Derivative Theories of Gravity II. Asymptotically Lifshitz Black Holes

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## Abstract

As a continuation of a previous work, here we examine the admittance of Birkhoff's theorem in a class of higher derivative theories of gravity. This class is contained in a larger class of theories which are characterized by the property that the trace of the field equations are of second order in the metric. The action representing these theories are given by a sum of higher curvature terms. Moreover the terms of a fixed order  $k$  in the curvature are constructed by taking a complete contraction of  $k$  conformal tensors. The general spherically (hyperbolic or plane) symmetric solution is then given by a static asymptotically Lifshitz black hole with the dynamical exponent equal to the spacetime dimensions. However, theories which are homogeneous in the curvature (i.e., of fixed order  $k$ ) possess additional symmetry which manifests as an arbitrary conformal factor in the general solution. So, these theories are analyzed separately and have been further divided into two classes depending on the order and the spacetime dimensions.

# 1 Introduction

In [1], the authors had presented a class of higher derivative theories of gravity which admits Birkhoff's theorem in vacuum<sup>1</sup>. These theories belong to a bigger class of theories which are characterized by second order traced field equations [3]. We had shown that there is a subclass of these theories whose field equations are generically of fourth order but when evaluated on a spherically (hyperbolic or plane) symmetric spacetimes reduce to second order thereby rendering the admittance of Birkhoff's theorem i.e., the corresponding solution is isometric to the static solution. Moreover the field equations and the solutions have a similar structure to those of Lovelock theories which are natural generalizations of Einstein's theory in higher dimensions [4]<sup>2</sup>. In the present work, as a continuation of the previous one, we show that there is another subclass of this bigger class of theories which admit Birkhoff's theorem. In this subclass, the scalar invariants of a fixed order  $k$  constituting the Lagrangian transform covariantly under conformal rescalings of the metric. Such Lagrangians are constructed by taking linear combination of complete contractions of the conformal tensor

$$C_{ab}{}^{cd} = R_{ab}{}^{cd} - \frac{4}{D-2} \delta_{[a}^{[c} R_{b]}^{d]} + \frac{2}{(D-2)(D-1)} \delta_{[a}^{[c} \delta_{b]}^{d]} R. \quad (1)$$

We will call such scalars *Weyl invariants*  $W^{(k)}$ , where the *order*  $k$  is the number of conformal tensors constituting the scalar. Then the action is expressed as a sum of terms, each of a fixed order  $k$  and is given by

$$I^{(k)} = \int d^D x \sqrt{-g} \sum_i^{N_D^{(k)}} \alpha_i^{(k)} W_i^{(k)}, \quad (2)$$

where  $N_D^{(k)}$  is the number of independent Weyl invariants  $W_i^{(k)}$  of order  $k$  in  $D$  dimensions. Before we prove the Birkhoff's theorem in the general class of theories given by an action containing terms of different order  $k$ , we shall examine the field equations and their solutions in theories with fixed order  $k$ . Such an analysis requires to be classified into two separate cases depending on the spacetime dimension  $D$  and the order  $k$ . They are

- $D = 2k$ : In this case, the action is invariant under local conformal transformations and hence the field equations and their solutions have a conformal symmetry.
- $D \neq 2k$ : In this case, the trace of the field equations is related to the Lagrangian by

$$\mathcal{G}_a^{(k)a} = \left(k - \frac{D}{2}\right) \mathcal{L}, \quad (3)$$

which implies that in vacuum the Lagrangian vanishes on its corresponding solutions.

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<sup>1</sup>This is a generalization of a theory cubic in curvature presented in [2] to arbitrary order.

<sup>2</sup>Birkhoff's theorem in Lovelock gravity was proved in [5]

In the next section, we explicitly evaluate the field equations on a spherically (hyperbolic or plane) symmetric spacetime ansatz. In section 3, we show that the theories of fixed order  $k$  admit a Birkhoff's theorem in a slightly weaker sense. As stated before, the analysis requires to be separated in two different cases depending on the spacetime dimensions. In section 4, we take up the general non-homogeneous action containing terms of different orders and prove the admittance of Birkhoff's theorem where the corresponding solution is an asymptotically Lifshitz black hole. Finally, in section 5 we shall summarize our results and their implications and mention some possible future directions of study.

## 2 Spherically (hyperbolic or plane) symmetric spacetimes

Consider the general spherically (plane or hyperbolic) symmetric spacetimes given by the following line element

$$ds^2 = \tilde{g}_{ij}(x)dx^i dx^j + e^{2\lambda(x)}d\Sigma_\gamma^2, \quad (4)$$

where  $d\Sigma_\gamma^2 = \hat{g}_{\alpha\beta}(y)dy^\alpha dy^\beta$  is the line element of a  $(D-2)$ -dimensional space of constant curvature  $\gamma$ . Let  $\tilde{\nabla}$  be the Levi-Civita connection on the two-dimensional space orthogonal to the constant curvature space and  $\tilde{R}$  be the corresponding scalar curvature. Then the nontrivial components of the Riemann curvature tensor and the conformal tensor are given by

$$R_{jl}{}^{ik} = \frac{1}{2}\tilde{R}\delta_{jl}^{ik}, \quad R_{\nu\rho}{}^{\mu\lambda} = \tilde{\mathcal{B}}\delta_{\nu\rho}^{\mu\lambda}, \quad R_{j\nu}{}^{i\mu} = -\tilde{\mathcal{A}}_j^i\delta_\nu^\mu, \quad (5)$$

$$C_{jl}{}^{ik} = \frac{(D-3)\tilde{S}}{2(D-1)}\delta_{jl}^{ik}, \quad C_{\nu\rho}{}^{\mu\lambda} = \frac{\tilde{S}}{(D-1)(D-2)}\delta_{\nu\rho}^{\mu\lambda}, \quad C_{j\nu}{}^{i\mu} = -\frac{(D-3)\tilde{S}}{2(D-1)(D-2)}\delta_j^i\delta_\nu^\mu, \quad (6)$$

where

$$\tilde{\mathcal{B}} = \gamma e^{-2\lambda} - (\tilde{\nabla}_m \lambda)(\tilde{\nabla}^m \lambda), \quad (7)$$

$$\tilde{\mathcal{A}}_j^i = \tilde{\nabla}^i \tilde{\nabla}_j \lambda + (\tilde{\nabla}^i \lambda)(\tilde{\nabla}_j \lambda), \quad (8)$$

$$\text{and } \tilde{S} = \tilde{R} + 2\tilde{\nabla}_k \tilde{\nabla}^k \lambda + 2\gamma e^{-2\lambda}. \quad (9)$$

Note that since all the components of the conformal tensor are a mere multiple of the function  $\tilde{S}$ , each of the conformal densities  $W_i^{(k)}$ 's evaluated on the metric (4) are proportional to  $\tilde{S}^k$ . Let  $W_m^{(k)} = \omega_m(D, k)\tilde{S}^k$ . Then the field equations for the action (2) evaluated on the metric ansatz (4) are given

by

$$\mathcal{G}^{(k)i}_j = k \left( \sum_{m=1}^{N_D^{(k)}} \alpha_m^{(k)} \omega_m(D, k) \right) \tilde{\mathcal{P}}_j^i(\tilde{S}^{k-1}) = k \alpha^{(k)} \tilde{\mathcal{P}}_j^i(\tilde{S}^{k-1}), \quad (10)$$

$$\mathcal{G}^{(k)\alpha}_\beta = k \left( \sum_{m=1}^{N_D^{(k)}} \alpha_m^{(k)} \omega_m(D, k) \right) \delta_\beta^\alpha \tilde{\mathcal{Q}}(\tilde{S}^{k-1}) = k \alpha^{(k)} \delta_\beta^\alpha \tilde{\mathcal{Q}}(\tilde{S}^{k-1}), \quad (11)$$

$$\mathcal{G}^{(k)i}_\alpha = \mathcal{G}^{(k)\alpha}_i = 0, \quad (12)$$

where  $\tilde{\mathcal{P}}_j^i$  and  $\tilde{\mathcal{Q}}$  are two (related) second order linear differential operators defined on the two-dimensional space orthogonal to the constant curvature base manifold given by

$$\begin{aligned} \tilde{\mathcal{P}}_j^i = & \left[ \delta_j^i \left( \frac{\tilde{R}}{2} + (D-1) \tilde{\nabla}_k \tilde{\nabla}^k \lambda + (D-2)(D-1) \tilde{\nabla}_k \lambda \tilde{\nabla}^k \lambda + \tilde{\nabla}_k \tilde{\nabla}^k + (2D-3) \tilde{\nabla}_k \lambda \tilde{\nabla}^k - \frac{\tilde{S}}{2k} \right) \right. \\ & \left. - (D-2)(\tilde{\nabla}^i \tilde{\nabla}_j \lambda + D \tilde{\nabla}^i \lambda \tilde{\nabla}_j \lambda) - \tilde{\nabla}^i \tilde{\nabla}_j - (D-1)(\tilde{\nabla}^i \lambda \tilde{\nabla}_j + \tilde{\nabla}_j \lambda \tilde{\nabla}^i) \right], \end{aligned} \quad (13)$$

$$\tilde{\mathcal{Q}} = -\frac{1}{D-2} \left[ \tilde{\mathcal{P}}_i^i - \tilde{S} \left( 1 - \frac{D}{2k} \right) \right]. \quad (14)$$

Note that (14) implies that for  $D = 2k$ , the trace of the field equations vanish identically, as should be the case for any conformally invariant theory.

### 3 Birkhoff's theorem in theories of fixed order $k$

We now prove a weaker version of Birkhoff's theorem for theories represented by the action (2). We present the proof for the cases  $D \neq 2k$  and  $D = 2k$  separately. In the following, we will assume

$$\alpha^{(k)} := \sum_{m=1}^{N_D^{(k)}} \alpha_m^{(k)} \omega_m(D, k) \neq 0. \quad (15)$$

If  $\alpha^{(k)} = 0$ , then any metric of the form (4) satisfies all the field equations.

#### 3.1 $D \neq 2k$

In this case, as explained earlier, the Lagrangian vanishes on its corresponding solutions. When evaluated on the spherically symmetric ansatz (4), this implies

$$\tilde{S} = \tilde{R} + 2\tilde{\nabla}_k \tilde{\nabla}^k \lambda + 2\gamma e^{-2\lambda} = 0 \quad (16)$$

Now, if we perform a conformal transformation in the 2-dimensional space as  $\tilde{g}_{ij} \rightarrow e^{2\lambda(x)} \tilde{g}_{ij}$ , then the above equation takes the form  $e^{-2\lambda}(\tilde{R} + 2\gamma) = 0$ . This in turn implies that the *new* two dimensional

metric  $\tilde{g}_{ij}$  is of constant curvature which always admits a non-null Killing vector. Adapting to these coordinates the solution can be written as

$$ds^2 = e^{2\lambda(t,\rho)} \left[ \Omega(\rho) (-dt^2 + d\rho^2) + d\Sigma_\gamma^2 \right] , \quad (17)$$

where  $\Omega(\rho)$  satisfies the equation

$$-\frac{1}{\Omega} \frac{d}{d\rho} \left( \frac{\Omega'}{\Omega} \right) + 2\gamma = 0 . \quad (18)$$

which can be integrated to obtain

$$\Omega(\rho) = \begin{cases} C_1 \cos^{-2}(\sqrt{C_1\gamma}(\rho + C_2)) & \text{when } \gamma \neq 0 \\ C_2 e^{C_1\rho} & \text{when } \gamma = 0 \end{cases} , \quad (19)$$

where  $C_1$  and  $C_2$  are integration constants. By a further redefinition of the coordinates

$$\frac{1}{r} = \begin{cases} -\frac{b}{2\gamma} + \sqrt{\frac{C_1}{\gamma}} \tan(\sqrt{C_1\gamma}(\rho + C_2)) & \text{when } \gamma \neq 0 \\ -b + \frac{C_2}{C_1} e^{C_1\rho} & \text{when } \gamma = 0 \end{cases} , \quad (20)$$

one can rewrite the metric in Schwarzschild-like coordinates as

$$ds^2 = e^{2\tilde{\lambda}(t,r)} \left[ -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_\gamma^2 \right] , \quad (21)$$

where  $f(r) = ar^2 + br + \gamma$  and the constants  $a, b$  are related to  $C_1$  and  $C_2$ . This metric is conformally flat and may represent an asymptotically locally flat or (A)dS black hole with a Cauchy horizon.

### 3.2 $D = 2k$

In this case, as explained earlier, the action and the corresponding field equations have a conformal symmetry. We exploit this symmetry to set  $\lambda = 0$ . Equation (10) then gives

$$\left[ \delta_j^i \left( \frac{\tilde{R}}{2} + \square - \frac{\tilde{S}}{2k} \right) - \tilde{\nabla}^i \tilde{\nabla}_j \right] \tilde{S}^{k-1} = 0 . \quad (22)$$

Note that equation (11) is then manifestly satisfied. Taking the trace of the above equation we obtain

$$\left( \tilde{R} - \frac{\tilde{S}}{k} \right) \tilde{S}^{k-1} = -\tilde{\square} \tilde{S}^{k-1} . \quad (23)$$

Plugging this back into equation (22) we get

$$\left( \delta_j^i \square - 2\tilde{\nabla}^i \tilde{\nabla}_j \right) \tilde{S}^{k-1} = 0 . \quad (24)$$

Contracting the above equation by the two dimensional Levi-Civita tensor  $\epsilon_{ki}$  and then symmetrizing the indices  $(j, k)$  we obtain

$$\tilde{\nabla}_{(j}\epsilon_{k)i}\tilde{\nabla}^i\tilde{S}^{k-1}=0. \quad (25)$$

This implies that the vector  $\tilde{\xi}_k = \epsilon_{ki}\tilde{\nabla}^i\tilde{S}^{k-1}$  satisfies the Killing equation  $\tilde{\nabla}_{(i}\tilde{\xi}_{k)} = 0$ . We now show that if  $\tilde{\xi}_k$  is a null Killing vector then the two dimensional metric  $\tilde{g}_{ij}$  is of constant curvature.

$$\begin{aligned} & \tilde{\xi}^k\tilde{\xi}_k = 0 \\ \Rightarrow & (\tilde{\nabla}^i\tilde{S})(\tilde{\nabla}_i\tilde{S}) = 0 \\ \Rightarrow & (\tilde{\nabla}_j\tilde{\nabla}^i\tilde{S})(\tilde{\nabla}_i\tilde{S}) = 0 \\ \Rightarrow & (\tilde{\square}\tilde{S}^{k-1})(\tilde{\nabla}_i\tilde{S}) = 0 && \text{using (24)} \\ \Rightarrow & \tilde{\square}\tilde{S}^{k-1} = 0 \\ \Rightarrow & \tilde{S} = 0 \quad \text{or} \quad \frac{2k\gamma}{k-1} && \text{using (23).} \end{aligned} \quad (26)$$

Therefore, in case  $\tilde{\xi}_k$  is a null Killing vector, then the metric is of constant curvature which in turn implies that the metric must admit at least one non-null Killing vector. So we could again adapt to these coordinates and take the following metric ansatz as previously

$$ds^2 = \Omega(\rho) [-dt^2 + d\rho^2] + d\Sigma_\gamma^2. \quad (27)$$

However, the field equations are not integrable in these coordinates. So instead we choose the following metric ansatz in the Schwarzschild coordinates

$$ds^2 = \frac{1}{r^2} \left[ -f(r)dt^2 + \frac{dr^2}{f(r)} \right] + d\Sigma_\gamma^2 \quad (28)$$

The Ricci scalar of the two-dimensional subspace is then given by

$$\tilde{R} = -r^3 \frac{d^2}{dr^2} \left( \frac{f(r)}{r} \right) \quad (29)$$

Equation (24) then gives

$$\tilde{\nabla}^t\tilde{\nabla}_t\tilde{S}^{k-1} = \tilde{\nabla}^\rho\tilde{\nabla}_\rho\tilde{S}^{k-1} \quad (30)$$

which can be integrated to obtain

$$\tilde{S} = \tilde{R} + 2\gamma = \left( \frac{c}{r} + d \right)^{\frac{1}{k-1}}. \quad (31)$$

Assuming  $c \neq 0$ , we can now use the expression (29) to further integrate the above equation and get

$$f(r) = ar^2 + br + \gamma - \frac{(k-1)^2}{c^2 k(2k-1)} r^2 \left( \frac{c}{r} + d \right)^{\frac{2k-1}{k-1}}. \quad (32)$$

Substituting this in equation (23), we find a further constraint among the integration constants given by

$$bc = 2\gamma d. \quad (33)$$

Hence, redefining the constants  $c$  and  $b$ , we can express the metric function as

$$f(r) = ar^2 + 2\gamma br + \gamma + cr^2 \left( \frac{1}{r} + b \right)^{\frac{2k-1}{k-1}}. \quad (34)$$

The constant  $c \neq 0$  can be set to 1 without any loss of generality. This can be seen by further redefining the constants as  $c = c'^{\frac{2k-1}{k-1}}$ ,  $bc' = b'$  and  $a = a'$  followed by the coordinate transformations  $r \rightarrow c'r$ ,  $t \rightarrow t/c'$  and relabeling the curvature of the *new*  $(D-2)$ -dimensional space  $\gamma' = \gamma/c'^2$  and finally removing all the primes. However, if  $c = 0$ , then the general metric function satisfying the field equations is given by

$$f(r) = ar^2 + br + d, \quad \text{where} \quad d = \gamma \quad \text{or} \quad -\frac{\gamma}{k-1}. \quad (35)$$

Therefore the general spherically symmetric solution of the theory given by the action (2) in dimensions  $D = 2k$  is given by the metric (21) with the function  $f(r)$  given by (34) or (35). The Birkhoff's theorem and the corresponding solution for  $k = 2$  was obtained in [6] for  $\gamma=1$  and in [7] for arbitrary  $\gamma$ .

## 4 Birkhoff's theorem for the general action

Now we shall prove a Birkhoff's theorem when the action is an arbitrary linear combination of terms of different orders. We also include a cosmological constant term. The action is then given by

$$I = \int d^D x \sqrt{-g} \left( \alpha^{(0)} + \sum_k \sum_i^{N_D^{(k)}} \alpha_i^{(k)} W_i^{(k)} \right). \quad (36)$$

The field equations of the above action evaluated on the metric ansatz (4) are then given by

$$\mathcal{G}_a^b = -\frac{1}{2} \alpha^{(0)} \delta_a^b + \sum_k \mathcal{G}^{(k)b}_a \quad (37)$$

Using the components of the field equations given by (10), (11) and (14) we then find the following polynomial equation in  $\tilde{S}$ .

$$\mathcal{G}_a^a = -\frac{D}{2}\alpha^{(0)} + \sum_k k\alpha^{(k)} \left(1 - \frac{D}{2k}\right) \tilde{S}^k = 0. \quad (38)$$

Let  $\beta \neq 0$  be a real root of the above polynomial. We next choose the coordinates  $(t, r)$  on the two-dimensional subspace such that the coordinate  $r = e^\lambda$  is spacelike and the metric (4) takes the form

$$ds^2 = -f(r, t)dt^2 + \frac{dr^2}{g(r, t)} + r^2 d\Sigma_\gamma^2 \quad (39)$$

The  $(t, r)$  component of the field equations then implies that  $g = g(r)$ . Using this we solve the difference of  $(t, t)$  and  $(r, r)$  components of the field equations. This implies

$$f(r, t) = \kappa(t)r^{2(D-1)}g(r) \quad (40)$$

where the arbitrary function  $\kappa(t)$  can be absorbed by redefining the coordinate  $t$  and thus we can take  $f = f(r) = r^{2(D-1)}g(r)$ . Next we use this and (9) to integrate the equation  $\tilde{S} = \beta$  and obtain the following form for the function  $g(r)$ .

$$g(r) = \frac{a}{r^{2(D-2)}} + \frac{b}{r^{D-2}} - \frac{\beta}{2D(D-1)}r^2 + \frac{\gamma}{(D-2)^2} \quad (41)$$

where  $a$  and  $b$  are integration constants. Finally, we obtain the sum of the  $(t, t)$  and  $(r, r)$  component of the field equations which gives

$$-\alpha^{(0)} + \sum_k k\alpha^{(k)}\beta^{k-1} \left[ \frac{2}{D}\beta \left(1 - \frac{D}{2k}\right) + (D-2)^2 \frac{b}{r^D} \right] = 0 \quad (42)$$

However, since  $\beta$  is a solution of the equation (38), the above equation implies  $b = 0$  unless  $\sum_k k\alpha^{(k)}\beta^{k-1} = 0$ . Thus the general spherically (hyperbolic or plane) symmetric solution of the theory given by the action (36) is

$$ds^2 = -\frac{r^{2D}}{l^{2D}}h(r)dt^2 + \frac{dr^2}{\frac{r^2}{l^2}h(r)} + r^2 d\Sigma_\gamma^2 \quad (43)$$

where the function  $h(r)$  is given by

$$h(r) = 1 + \frac{a}{r^{2(D-1)}} + \frac{\gamma l^2}{(D-2)^2 r^2}, \quad \text{when } \sum_k k\alpha^{(k)}\beta^{k-1} \neq 0 \quad (44)$$

$$= 1 + \frac{a}{r^{2(D-1)}} + \frac{b}{r^D} + \frac{\gamma l^2}{(D-2)^2 r^2}, \quad \text{when } \sum_k k\alpha^{(k)}\beta^{k-1} = 0 \quad (45)$$



where  $a$  and  $b$  are new integration constants,  $t$  has been rescaled and  $\beta = -\frac{2D(D-1)}{l^2}$  is a non-zero real solution of (38). Note that when  $\alpha^{(0)} = 0$  and  $\beta = 0$ , the general spherically (hyperbolic or plane) symmetric solution is given by (21). The metric (43) is a static asymptotically Lifshitz spacetime with the dynamical exponent equal to the spacetime dimensions and represents a black hole for negative values of  $a$ .

## 5 Conclusions

In summary, we have proved the admittance of Birkhoff's theorem in a particular class of higher derivative theories. The action representing these theories consists of invariants which are constructed by taking complete contractions of a number of conformal tensors. We have shown that in general the spherically (hyperbolic or plane) symmetric solution of such a theory is given by an asymptotically Lifshitz spacetime whose dynamical exponent is equal to the spacetime dimensions. However, in the particular cases of theories homogeneous in the curvature, there is an additional symmetry that manifests as an arbitrary conformal factor in the general solution. These have been further classified into two different cases based on the spacetime dimension  $D$  and the order  $k$ .

Let us now point out some of the important differences between the theories considered in [1] and here. In [1] we had considered a class of higher derivative theories whose field equations when evaluated on spherically (hyperbolic or plane) symmetric spacetimes reduce to second order. Moreover, the structure of these second order equations is the same as those of Lovelock theories. In contrast, here the theories under study yield field equations which after evaluating on the spherically (hyperbolic or plane) symmetric ansatz are still of fourth order. Consequently the general solution has a different form than that of a Lovelock theory. This answers some of the questions raised in [1]. The analysis here explicitly shows that for Birkhoff's theorem to hold it is not necessary that the field equations evaluated on spherically (hyperbolic or plane) symmetric spacetimes reduce to second order. Moreover, it shows that the admittance of Birkhoff's theorem does not imply that the corresponding solutions have the same form as those of Lovelock theories.

Now a few comments are in order. The homogeneous theories of order  $k$  in  $D = 2k$  dimensions are conformally invariant. The simplest case of  $k = 2$  in four dimensions is known as conformal gravity, originally introduced by Bach in [8] and has been a subject of interest for various reasons. Recently it has been shown in [9] that by imposing a simple Neumann boundary condition on the metric, conformal gravity can be shown to be equivalent to Einstein gravity with a cosmological constant. This is possible because the solutions of Einstein's gravity are also solutions of conformal gravity. It is interesting to note that setting  $a = b = 0$  in (34) one obtains the spherically symmetric solution of pure Lovelock theory of order  $k - 1$ .

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